

Discrete dynamical systems with $W(A_{m-1}^{(1)} \times A_{n-1}^{(1)})$ symmetry

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Abstract

We give a birational realization of affine Weyl group of type $A_{m-1}^{(1)} \times A_{n-1}^{(1)}$. We apply this representation to construct some discrete integrable systems and discrete Painlevé equations. Our construction has a combinatorial counterpart through the ultra-discretization procedure.

1 Introduction

It is known that continuous and discrete Painlevé equations admit affine Weyl group symmetry. These symmetries are realized as birational Bäcklund transformations and can be described in universal way for general root systems.[4] On the other hand, in recent studies on (ultra-)discrete integrable systems, certain affine Weyl group representation has been obtained.[10] This representation is “tropical” (i.e. given in terms of subtraction-free birational mapping) and has a combinatorial counterpart through the ultra-discretization procedure. The aim of this paper is to reveal the relation between these two affine Weyl group representations. In fact, we show that they are essentially the same.

In section 2, we recall the definition of birational transformations π, r_0, \dots, r_m and ρ, s_0, \dots, s_n acting on mn variables x_{ij} ($i = 1, \dots, m, j = 1, \dots, n$). The main result (Theorem 1.1) states that these transformations give a representation of affine Weyl group of type $A_{m-1}^{(1)} \times A_{n-1}^{(1)}$. We give the proof of Theorem 1.1 in section 3. In section 4, we describe the discrete dynamical systems arising from our affine Weyl group representation. The systems can be viewed as a version of discrete Toda equation and its generalizations. We also discuss the relation with q -Painlevé equation studied in [1].

2 Affine Weyl group actions

Let m, n be positive integers and $K = \mathbf{C}(x)$ be the field of rational functions on mn variables x_{ij} , $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$. We extend the indices i, j of x_{ij} for $i, j \in \mathbf{Z}$ by the condition $x_{i+m,j} = x_{i,j+n} = x_{ij}$.

Define algebra automorphisms π, ρ, r_i and s_j ($i \in \mathbf{Z}/m\mathbf{Z}$, $j \in \mathbf{Z}/n\mathbf{Z}$) on the field K as follows:

$$\pi(x_{ij}) = x_{i+1,j}, \quad \rho(x_{i,j}) = x_{i,j+1}, \quad (1)$$

$$r_i(x_{ij}) = x_{i+1,j} \frac{P_{i,j-1}}{P_{ij}}, \quad r_i(x_{i+1,j}) = x_{ij} \frac{P_{ij}}{P_{i,j-1}}, \quad (2)$$

$$r_k(x_{ij}) = x_{ij}, \quad (k \neq i, i+1), \quad (3)$$

$$s_j(x_{ij}) = x_{i,j+1} \frac{Q_{i-1,j}}{Q_{ij}}, \quad s_j(x_{i,j+1}) = x_{ij} \frac{Q_{ij}}{Q_{i-1,j}}, \quad (4)$$

$$s_k(x_{ij}) = x_{ij}, \quad (k \neq j, j+1), \quad (5)$$

where

$$P_{ij} = \sum_{a=1}^n \left(\prod_{k=1}^{a-1} x_{i,j+k} \prod_{k=a+1}^n x_{i+1,j+k} \right), \quad (6)$$

$$Q_{ij} = \sum_{a=1}^m \left(\prod_{k=1}^{a-1} x_{i+k,j} \prod_{k=a+1}^m x_{i+k,j+1} \right). \quad (7)$$

The following is the main result of this note.

Theorem 2.1 $\langle \pi, r_0, r_1, \dots, r_{m-1} \rangle$ and $\langle \rho, s_0, s_1, \dots, s_{n-1} \rangle$ generate the extended affine Weyl group $\widetilde{W}(A_{m-1}^{(1)})$ and $\widetilde{W}(A_{n-1}^{(1)})$. Moreover these two actions $\widetilde{W}(A_{m-1}^{(1)})$ and $\widetilde{W}(A_{n-1}^{(1)})$ mutually commute.

In our previous work [10], the first part of this theorem has been proved and the second part was conjectured. We will prove Theorem 2.1 in the next section.

Example 2.2 For the case of $(m, n) = (2, 3)$. Let us write the x_{ij} variables as

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \quad (8)$$

Then the actions $r_1, r_0, \pi, s_1, s_2, s_0$ and ρ are given as follows.

$$\begin{aligned} r_1(x_1) &= y_1 \frac{x_1 x_2 + x_1 y_3 + y_2 y_3}{x_2 x_3 + x_2 y_1 + y_3 y_1}, & r_1(y_1) &= x_1 \frac{x_2 x_3 + x_2 y_1 + y_3 y_1}{x_1 x_2 + x_1 y_3 + y_2 y_3}, \\ r_1(x_2) &= y_2 \frac{x_2 x_3 + x_2 y_1 + y_3 y_1}{x_3 x_1 + x_2 y_2 + y_1 y_2}, & r_1(y_2) &= x_2 \frac{x_3 x_1 + x_2 y_2 + y_1 y_2}{x_2 x_3 + x_2 y_1 + y_3 y_1}, \\ r_1(x_3) &= y_3 \frac{x_3 x_1 + x_2 y_2 + y_1 y_2}{x_1 x_2 + x_1 y_3 + y_2 y_3}, & r_1(y_3) &= x_3 \frac{x_1 x_2 + x_1 y_3 + y_2 y_3}{x_3 x_1 + x_2 y_2 + y_1 y_2}, \end{aligned} \quad (9)$$

$\pi(x_i) = y_i, \pi(y_i) = x_i, r_0 = \pi r_1 \pi$ and

$$\begin{aligned} s_1(x_1) &= x_2 \frac{x_1 + y_2}{x_2 + y_1}, & s_1(x_2) &= x_1 \frac{x_2 + y_1}{x_1 + y_2}, & s_1(x_3) &= x_3, \\ s_1(y_1) &= y_2 \frac{x_2 + y_1}{x_1 + y_2}, & s_1(y_2) &= y_1 \frac{x_1 + y_2}{x_2 + y_1}, & s_1(y_3) &= y_3, \end{aligned} \quad (10)$$

$\rho(x_i) = x_{i+1}, \rho(y_i) = y_{i+1}, i \in \mathbf{Z}/(3\mathbf{Z}), s_2 = \rho s_1 \rho^{-1}$ and $s_0 = \rho^2 s_1 \rho^{-2}$.

3 Proof of Theorem 2.1

The first part (affine Weyl group relations): Though this part of the theorem has already been proved in [10] by direct computation, we give more conceptual proof here. We prove that s_i ($i = 1, \dots, n-1$) satisfy the same relations as those for the adjacent permutations $\sigma_i = (i, i+1) \in S_n$. The relations containing s_0 follow from the cyclic symmetry of the definition (4,5,7).

Using the identities,

$$x_{i,l+1} Q_{i-1,l} - x_{il} Q_{il} = k_{l+1} - k_l, \quad k_l = \prod_{i=1}^m x_{il}. \quad (11)$$

we see that the automorphisms s_j satisfy the relations,

$$x_i y_i = x'_i y'_i, \quad x_i + y_{i+1} = x'_i + y'_{i+1}, \quad (12)$$

where $x_i = x_{ij}, y_i = x_{i,j+1}, x'_i = s_j(x_{ij})$ and $y'_i = s_j(x_{i,j+1})$. The relations (12) can be written in matrix form as

$$A(x)A(y) = A(x')A(y'), \quad (13)$$

where

$$A(x) = \begin{pmatrix} x_1 & 1 & & & \\ & x_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & x_{m-1} & 1 \\ z & & & & x_m \end{pmatrix}. \quad (14)$$

The relations (12) essentially characterize the transformation s_j . More generally we have the following,

Lemma 3.1 *For given mn variables $x_j = (x_{1j}, x_{2j}, \dots, x_{mj})$ and mn unknowns $x'_j = (x'_{1j}, x'_{2j}, \dots, x'_{mj})$, a system of algebraic equations*

$$A(x_1)A(x_2) \cdots A(x_n) = A(x'_1)A(x'_2) \cdots A(x'_n), \quad (15)$$

has $n!$ solutions. Each of the solution corresponds to a permutation $\sigma \in S_n$ and characterized by additional condition

$$x'_{1j}x'_{2j} \cdots x'_{mj} = x_{1\sigma(j)}x_{2\sigma(j)} \cdots x_{m\sigma(j)}. \quad (16)$$

Proof of Lemma 3.1. From the definition of the transformation s_j (4,5,7), it follows that

$$A(s_j(x_1)) \cdots A(s_j(x_n)) = A(x_1) \cdots A(x_n), \quad (17)$$

and

$$s_j(x_{1k} \cdots x_{mk}) = x_{1\sigma_j(k)} \cdots x_{m\sigma_j(k)}, \quad k = 1, \dots, n, \quad (18)$$

where $\sigma_j = (j, j+1)$. This implies that for any permutation σ given as a product $\sigma = \sigma_{i_1} \cdots \sigma_{i_k}$,

$$x'_{ij} = s_{i_1} \cdots s_{i_k}(x_{ij}), \quad (19)$$

is a solution of (15) and (16). Hence we see that there exist at least $n!$ solutions for (15).

Let us next show that such solution is unique, namely we prove that the number of solutions for (15) is at most $n!$. We discuss the case $n = 3$ for simplicity. Consider a system of algebraic equations for $3m$ unknown variables (x'_i, y'_i, z'_i) given by

$$A(x)A(y)A(z) = A(x')A(y')A(z'), \quad (20)$$

or

$$a_i = a'_i, \quad b_i = b'_i, \quad c_i = c'_i, \quad (21)$$

where

$$a_i = x_i + y_{i+1} + z_{i+2}, \quad b_i = x_i y_i + x_i z_{i+1} + y_{i+1} z_{i+1}, \quad c_i = x_i y_i z_i; \quad (22)$$

a'_i, b'_i and c'_i are defined similarly. By simple elimination of variables y'_i and z'_i , one has algebraic equations for x'_i

$$x'_{i-1}x'_i x'_{i+1} - a_{i-1}x'_i x'_{i+1} + b_i x'_{i+1} - c_{i+1} = 0, \quad (23)$$

or equivalently

$$x'_{i+1}\psi_i = R_{i+1}\psi_{i+1}, \quad R_{i+1} = \begin{bmatrix} 1 & & \\ c_{i+1} & -b_i & a_{i-1} \end{bmatrix}, \quad \psi_i = \begin{bmatrix} 1 \\ x'_i \\ x'_{i-1}x'_i \end{bmatrix}. \quad (24)$$

Due to the periodicity $x'_{i+m} = x'_i$, we have

$$(x'_1 x'_2 \cdots x'_m) \psi_1 = (R_m \cdots R_2 R_1) \psi_1. \quad (25)$$

Hence, ψ_1 is one of the three eigenvectors of $R_m \cdots R_2 R_1$. Once ψ_1 is determined other ψ_i 's and x'_i 's are determined uniquely by using (24) repeatedly. Then we have the equation for y'_i 's and z'_i 's

$$A(x')^{-1} A(x) A(y) A(z) = A(y') A(z'). \quad (26)$$

By similar argument we see that this equation has two solutions. Hence we have at most $3!$ solutions for (20). ■

If $\sigma = \sigma_{i_1} \cdots \sigma_{i_k} = \sigma_{j_1} \cdots \sigma_{j_l}$ then

$$s_{i_1} \cdots s_{i_k} (x_{1r} \cdots x_{mr}) = x_{1\sigma(r)} \cdots x_{m\sigma(r)} = s_{j_1} \cdots s_{j_l} (x_{1r} \cdots x_{mr}), \quad (27)$$

and we have $s_{i_1} \cdots s_{i_k} = s_{j_1} \cdots s_{j_l}$ due to Lemma 3.1. This means that all the relations satisfied by σ_i 's also hold for s_i 's.

The second part (commutativity): Let us put $G_i(u) = 1 + uE_{i+1,i}$ for $i \in [1, m-1]$ and $G_0(u) = 1 + \frac{u}{z} E_{1,m}$. We also put

$$M = A(x_1) A(x_2) \cdots A(x_n), \quad (28)$$

where $x_j = (x_{1j}, \cdots, x_{mj})$. We remark that Lemma 3.1 also implies the invariance of M under the actions of s_j ($j = 1, 2, \dots, n-1$). The action of r_k on M is described as follows.

Lemma 3.2

$$G_k(u) M = r_k(M) G_k(u), \quad u = \frac{h_k - h_{k+1}}{P_{k,0}}. \quad (29)$$

Proof. Putting

$$u_j = \frac{h_k - h_{k+1}}{P_{kj}} \quad (j = 0, 1, \dots, n), \quad (30)$$

the action of r_k can be rewritten in the form

$$\begin{aligned} r_k(x_{kj}) &= x_{kj} - u_j, & r_k(x_{k+1,j}) &= x_{k+1,j} + u_{j-1}, \\ r_k(x_{ij}) &= x_{ij} & (i \neq k, k+1), \end{aligned} \quad (31)$$

where we have used the identity

$$x_{k+1,j} P_{k,j-1} - x_{kj} P_{kj} = h_{k+1} - h_k, \quad h_k = \prod_{j=1}^n x_{kj}. \quad (32)$$

The relations above are equivalent to the matrix equation

$$G_k(u_{j-1})A(x_j) = A(r_k(x_j))G_k(u_j) \quad (j = 1, \dots, n). \quad (33)$$

Hence we have

$$G_k(u_0)A(x_1) \cdots A(x_n) = A(r_k(x_1)) \cdots A(r_k(x_n))G_k(u_n). \quad (34)$$

Due to the periodicity $u_0 = u_n (= u)$, we have

$$G_k(u)M = r_k(M)G_k(u), \quad u = \frac{h_k - h_{k+1}}{P_{k,0}}, \quad (35)$$

where $M = A(x_1) \cdots A(x_n)$. ■

We remark that the parameter u can also be determined from M by the formula

$$u = \frac{M_{kk} - M_{k+1,k+1}}{M_{k,k+1}} \Big|_{z=0}. \quad (36)$$

The commutativity of the actions of the two Weyl groups in Theorem 2.1 follows from Lemma 3.2, because the matrix M is invariant under the actions of s_j ($j = 1, 2, \dots, n-1$). The commutativity $r_i s_0 = s_0 r_i$ follows from $r_0 = \rho^{-1} s_1 \rho$ and $r_i \rho = \rho r_i$. Thus, the proof of Theorem 2.1 is completed.

In this proof, we have treated the actions s_i and r_j asymmetrically. However the argument can be applicable for both of them, because the roles of s_i and r_j can be exchanged with each other by the transposition of the matrix $X = (x_{ij})$.

4 Discrete dynamical systems

The affine Weyl group $\widetilde{W}(A_{m-1}^{(1)}) \times \widetilde{W}(A_{n-1}^{(1)})$ has a translation subgroup $\mathbf{Z}^{m-1} \times \mathbf{Z}^{n-1}$. If we consider a part of these translations as providing a discrete dynamical system, the commutant of them can be viewed as its Bäcklund transformations.

Example 4.1 When $m = 2$, the translation subgroup of $\widetilde{W}(A_1^{(1)})$ is generated by $T = \pi r_1$. Its explicit actions $T(x_i) = \bar{x}_i$ etc. are determined as the nontrivial solution of

$$\bar{x}_j \bar{y}_j = x_j y_j, \quad \bar{x}_j + \bar{y}_{j+1} = x_{j+1} + y_j, \quad (37)$$

where $x_j = x_{1j}$ and $y_j = x_{2j}$ ($j \in \mathbf{Z}/n\mathbf{Z}$). This system is a version of the *discrete Toda equation*. [8, 9] The explicit time evolution is given by

$$\bar{x}_j = x_j \frac{P_{j-1}}{P_j}, \quad \bar{y}_j = y_j \frac{P_j}{P_{j-1}}, \quad (38)$$

where

$$P_j = \sum_{a=1}^n \overbrace{y_{j+1} \cdots y_{j+a-1}}^{a-1} \overbrace{x_{j+a+1} \cdots x_{j+n}}^{n-a}. \quad (39)$$

The Weyl group $\widetilde{W}(A_{n-1}^{(1)})$ commutes with this discrete evolution T . The explicit actions are

$$\begin{aligned} s_j(x_j) &= x_{j+1} \frac{x_j + y_{j+1}}{x_{j+1} + y_j}, & s_j(y_j) &= y_{j+1} \frac{x_{j+1} + y_j}{x_j + y_{j+1}}, \\ s_j(x_{j+1}) &= x_j \frac{x_{j+1} + y_j}{x_j + y_{j+1}}, & s_j(y_{j+1}) &= y_j \frac{x_j + y_{j+1}}{x_{j+1} + y_j}, \end{aligned} \quad (40)$$

$$s_j(x_k) = x_k, \quad s_j(y_k) = y_k, \quad (k \neq j, j+1).$$

Example 4.2 When $m = 3$, we use the variables $x_j = x_{1j}$, $y_j = x_{2j}$ and $z_j = x_{3j}$. Then $\pi(x_j) = y_j$, $\pi(y_j) = z_j$, $\pi(z_j) = x_j$, $\rho(x_j) = x_{j+1}$, $\rho(y_j) = y_{j+1}$ and $\rho(z_j) = z_{j+1}$ ($j \in \mathbf{Z}/n\mathbf{Z}$). The translation subgroup of $\widetilde{W}(A_2^{(1)})$ is generated by

$$T_1 = \pi r_2 r_1, \quad T_2 = r_1 \pi r_2. \quad (41)$$

We describe the action of T_1 , then the action of $T_2 = \pi T_1 \pi^{-1}$ is given by the rotation. The actions $T(x_i) = \bar{x}_i$ etc. are determined from the system of algebraic equations

$$\begin{aligned} \bar{x}_j \bar{y}_j \bar{z}_j &= x_j y_j z_j, \\ \bar{x}_j \bar{y}_j + \bar{x}_j \bar{z}_{j+1} + \bar{y}_{j+1} \bar{z}_{j+1} &= y_j z_j + y_j x_{j+1} + z_{j+1} x_{j+1}, \\ \bar{x}_j + \bar{y}_{j+1} + \bar{z}_{j+2} &= y_j + z_{j+1} + x_{j+2}. \end{aligned} \quad (42)$$

as a unique rational solution such that

$$\bar{x}_1 \cdots \bar{x}_n = y_1 \cdots y_n, \quad \bar{y}_1 \cdots \bar{y}_n = z_1 \cdots z_n, \quad \bar{z}_1 \cdots \bar{z}_n = x_1 \cdots x_n. \quad (43)$$

The system (42) can be regarded as a generalization of the discrete Toda equation. The explicit formulas for the time evolution T_1 are given in the form

$$\bar{x}_j = x_j \frac{F_{j-1}}{F_j}, \quad \bar{y}_j = y_j \frac{G_{j-1} F_j}{G_j F_{j-1}}, \quad \bar{z}_j = z_j \frac{G_j}{G_{j-1}}. \quad (44)$$

Here F_j and G_j are polynomials in x, y, z such that $F_j = \rho^j(F_0)$, $G_j = \rho^j(G_0)$. When $n = 3$ for example, F_0 and G_0 are given explicitly as

$$\begin{aligned} F_0 &= x_1 x_2 x_3^2 + x_1 x_2 x_3 y_1 + x_2 x_3 y_1 y_2 + x_3 y_1 y_2 z_1 + x_1 x_2 x_3 z_2 \\ &\quad + y_1 y_2 z_1 z_2 + x_2 x_3 y_1 z_3 + x_3 y_1 z_1 z_3 + x_2 x_3 z_2 z_3, \\ G_0 &= x_2 x_3 + x_3 z_1 + z_1 z_2. \end{aligned} \quad (45)$$

We remark that the affine Weyl group $\widetilde{W}(A_{n-1}^{(1)})$ provides Bäcklund transformations for the time evolutions T_1, T_2 .

Our dynamical system has the following conserved quantities.

Proposition 4.3 *The characteristic polynomial of the matrix*

$$M = A(x_1) \cdots A(x_n) \quad (46)$$

is invariant under the actions of the affine Weyl group $\widetilde{W}(A_{m-1}^{(1)}) \times \widetilde{W}(A_{n-1}^{(1)})$.

In fact, by Lemma 3.1, the matrix M itself is invariant under the action of s_i . On the other hand, the action of each $r_j(M)$ is conjugate to M as we have seen in Lemma 3.2.

Example 4.4 For $(m, n) = (2, 3)$, we have the following invariants.¹

$$\begin{aligned} \det[A(x)A(y) + wI] &= w^3 + z^2 + w^2(x_1y_1 + x_2y_2 + x_3y_3) + \\ & z(x_1x_2x_3 + y_1y_2y_3) + w(x_1x_2y_1y_2 + x_2x_3y_2y_3 + x_3x_1y_3y_1) + \\ & zw(x_1 + x_2 + x_3 + y_1 + y_2 + y_3) + x_1x_2x_3y_1y_2y_3. \end{aligned} \quad (47)$$

Since the Weyl group actions are *tropical* [2] (i.e. subtraction-free birational mappings), there exists a combinatorial counterpart obtained by the ultra-discretization [5]

$$a \times b \rightarrow a + b, \quad a + b \rightarrow \min(a, b). \quad (48)$$

Corresponding combinatorial dynamical system can be viewed as a periodic version of the Box-Ball systems. Similarly to the original Box-Ball system [6] and their generalizations [7], this dynamical system also has soliton type solutions.

The transformations as in Example 4.1 are quite similar to those for the q -Painlevé equation q - P_{IV} and their Bäcklund transformations. In fact, the realization of the Weyl group as in (29), (36) is defined exactly in the same way as that of the Bäcklund transformations of the Painlevé equations [3] and their Lie-theoretic generalization studied in [4], Section 4. The q - P_{IV} can be viewed as a non-autonomous deformation of our integrable system for $(m, n) = (2, 3)$ of this paper.

In order to see this surprising coincidence more precisely, let us reformulate Example 4.1. Putting $x_j y_j = b_j$, $a_j = b_j / b_{j+1}$ and $x_j x_{j+1} = \varphi_j b_j$, we have

$$\bar{b}_j = b_j, \quad \frac{\varphi_j + 1}{\varphi_j + a_j} = \frac{x_{j+1}}{x_j}. \quad (49)$$

Eliminating the x_j variables, one obtain

$$(1 + \varphi_j) \left(1 + \frac{1}{\varphi_{j+1}}\right) = \left(1 + \frac{\overline{\varphi_{j+1}}}{a_{j+1}}\right) \left(1 + \frac{a_j}{\varphi_j}\right). \quad (50)$$

¹ By the $m \leftrightarrow n$ duality, the same invariants can also be obtained as the characteristic polynomial of a 2×2 matrix.

In terms of the variables a_j and φ_j , the time evolution (38) is rewritten in the form

$$\begin{aligned}\bar{a}_j &= a_j, \quad \bar{\varphi}_j = a_j \varphi_{j+1} \frac{g_{j+2}}{g_j}, \\ g_j &= 1 + \varphi_j + \varphi_j \varphi_{j+1} + \cdots + \varphi_j \cdots \varphi_{j+n-2}.\end{aligned}\tag{51}$$

The Weyl group $\widetilde{W}(A_{n-1}^{(1)})$ commutes with this discrete evolution. In terms of variables a_j and φ_j , the Bäcklund transformations (40) are given by

$$\begin{aligned}s_j(a_j) &= \frac{1}{a_j}, \quad s_j(a_{j+1}) = a_{j+1}a_j, \quad s_j(a_{i-1}) = a_{i-1}a_j, \\ s_j(\varphi_j) &= \frac{\varphi_j}{a_j}, \quad s_j(\varphi_{j+1}) = \varphi_{j+1} \frac{a_j + \varphi_j}{1 + \varphi_j}, \quad s_j(\varphi_{i-1}) = \varphi_{i-1} \frac{1 + \varphi_j}{1 + \varphi_j/a_j},\end{aligned}\tag{52}$$

and $s_j(a_k) = a_k$, $s_j(\varphi_k) = \varphi_k$ for $k \neq j, j \pm 1 \pmod{n}$.

In this setting, the parameters a_j are subject to the constraint

$$q \equiv a_0 a_1 \cdots a_{n-1} = 1.\tag{53}$$

However, the Weyl group representation in (52) and its commutativity with the evolution (51) survive also for $q \neq 1$. In this non-autonomous situation, the equation (51) gives a generalized q -Painlevé equation which has the $\widetilde{W}(A_{n-1}^{(1)})$ Bäcklund transformations (52); the case $n = 3$ recovers the q - P_{IV} equation [1]. The non-autonomous versions of the cases with $m \geq 3$ could be considered as a q -Painlevé hierarchy with multi-discrete times.

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